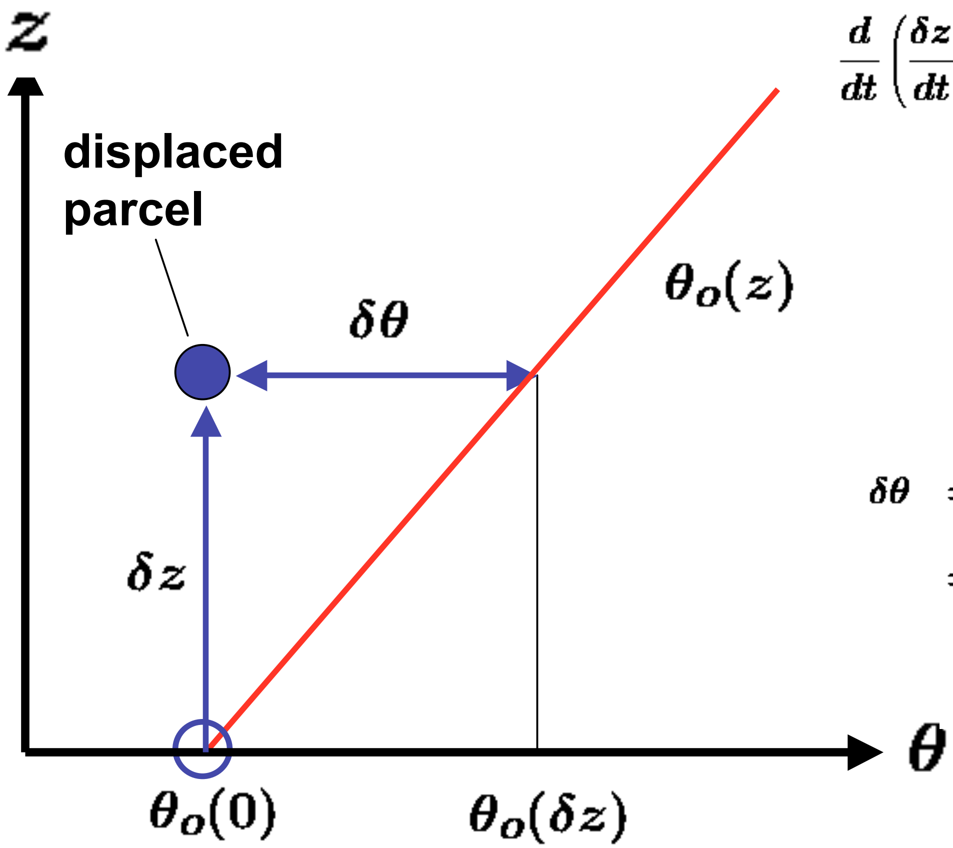
Part 2: Background Theory

Buoyancy oscillations



$$\frac{d}{dt}\left(\frac{\delta z}{dt}\right) = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$ (Newton's 2nd Law)

$$= -g - \frac{1}{\rho}\frac{dp_o}{dz}$$ (hydrostatic basic state)

$$= g\left(\frac{\rho_o - \rho}{\rho}\right)$$

$$= g\left(\frac{\theta - \theta_o}{\theta_o}\right)$$ (Ideal gas law and definition of potential temperature)

$$\delta\theta = \theta(\delta z) - \theta_o(\delta z) = \theta_o(0) - \theta_o(\delta z)$$ (potential temperature of parcel is conserved)

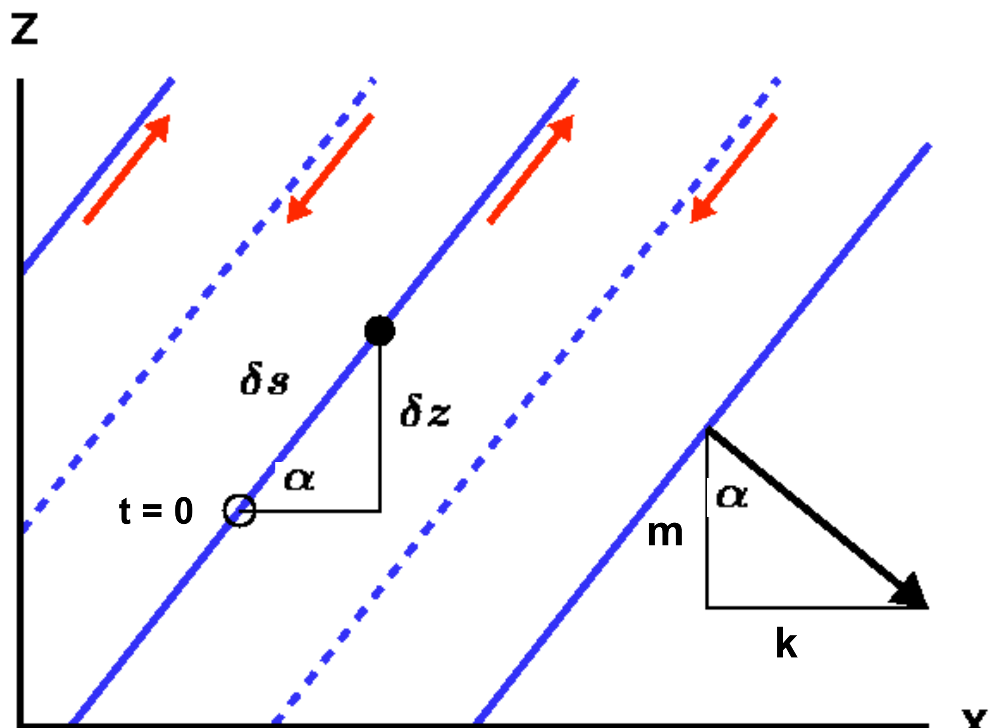
$$\theta_o(\delta z) \approx \theta_o(0) + \frac{d\theta_o}{dz}\delta z$$ (for small vertical displacement)

$$g\left(\frac{\theta - \theta_o}{\theta_o}\right) = -\left(\frac{g}{\theta_o}\frac{d\theta_o}{dz}\right)\delta z = -N^2\,\delta z$$ (buoyancy force)

$$\frac{d^2(\delta z)}{dt^2} + N^2(\delta z) = 0 \quad \rightarrow \quad \delta z = Re\left\{Ae^{iNt} + Be^{-iNt}\right\}$$

Parcel oscillates in the vertical with the buoyancy frequency N.

Internal gravity waves



Now consider the parcel of air displaced upward along a sloping surface:

Acceleration of parcel up the slope is

$$\frac{d^2(\delta s)}{dt^2} = \frac{d^2}{dt^2}\left(\frac{\delta z}{\sin\alpha}\right)$$

Component of buoyancy force along the slope is

$$-N^2 \delta z \sin\alpha$$

Newton's 2nd law gives

$$\frac{d^2}{dt^2}\left(\frac{\delta z}{\sin\alpha}\right) + N^2 \sin\alpha\,\delta z = 0$$

$$\Rightarrow \frac{d^2}{dt^2}\delta z + \omega^2\,\delta z = 0$$

$$\begin{aligned}\omega^2 &= N^2 \sin^2\alpha \\ &= N^2 \frac{k^2}{k^2+m^2} \approx \frac{N^2 k^2}{m^2}\end{aligned}$$

which is the dispersion relation for a hydrostatic gravity wave in the Boussinesq approximation.

Linear gravity wave equations

Equations governing hydrostatic gravity waves without rotation are:

$$\frac{\partial u'}{\partial t} + \bar{u}\frac{\partial u'}{\partial x} + \frac{\partial \Phi'}{\partial x} = 0$$

(zonal momentum equation)

$$\frac{\partial}{\partial t}\left(\frac{\partial \Phi'}{\partial z}\right) + \bar{u}\frac{\partial}{\partial x}\left(\frac{\partial \Phi'}{\partial z}\right) + N^2 w' = 0$$

(thermodynamic equation)

$$\frac{\partial u'}{\partial x} + \frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o w') = 0$$

(mass continuity equation)

Consider plane wave solutions:

$$f' = e^{z/2H}\mathrm{Re}\left\{\tilde{f}\, e^{i(kx+mz-\omega t)}\right\}$$

Resulting **dispersion relation** is:

$$\hat{\omega}^2 = \frac{N^2k^2}{m^2 + 1/(4H^2)}$$

$$\hat{\omega} = \omega - k\bar{u}$$ **(intrinsic frequency)**

Nonhydrostatic and rotational effects

• the previous example was for linear hydrostatic GWs without the effects of the Earth's rotation.

• a more general dispersion relation which accounts for nonhydrostatic and rotational effects is:

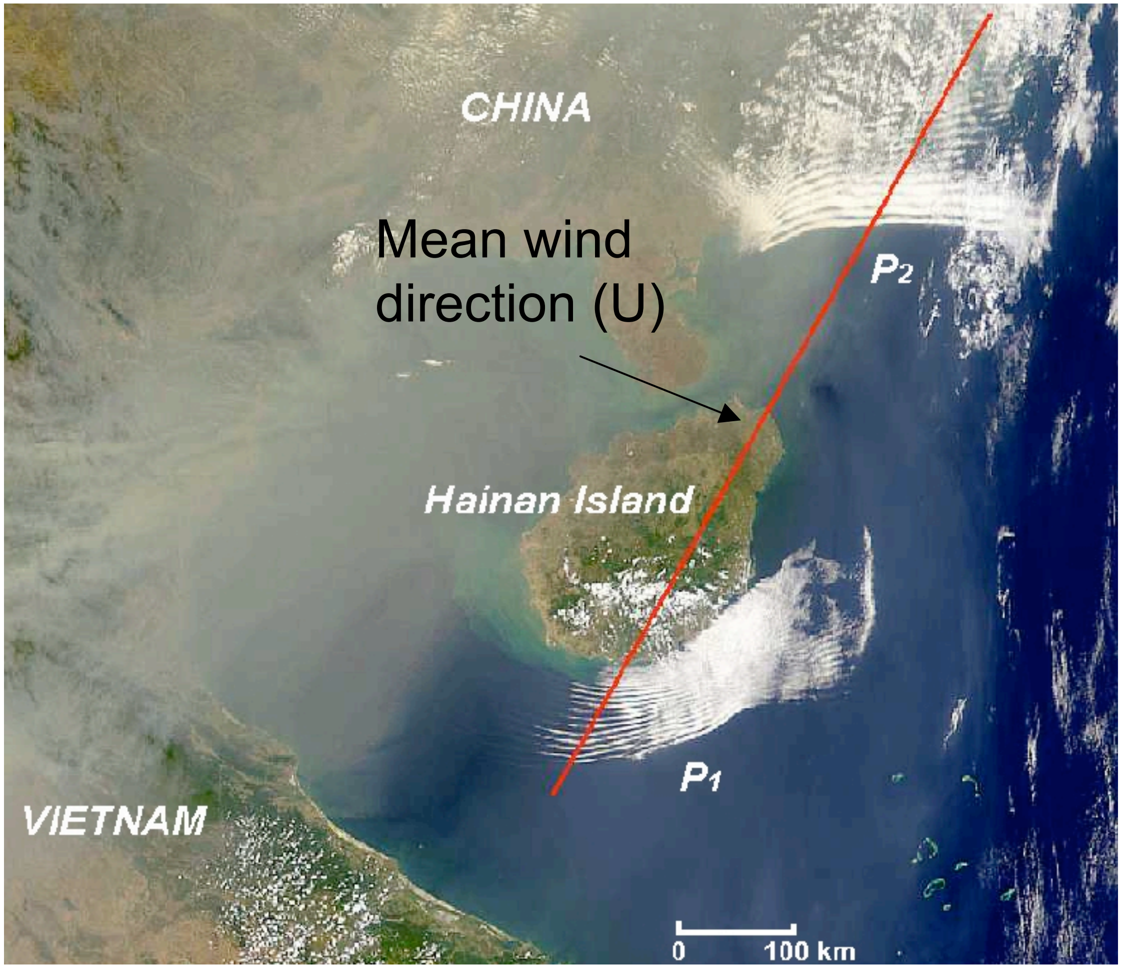
$$m^2 = \frac{(N^2 - \hat{\omega}^2)\, k^2}{\hat{\omega}^2 - f^2}$$

where f is the Coriolis parameter.

• a vertically propagating wave (m real) now requires that

$$|f| < |\hat{\omega}| < N$$

Vertically trapped mountain waves



- For a stationary mountain wave with

$$|\hat{\omega}| >> |f|$$

$$\hat{\omega} = -k\bar{u}$$

- Dispersion relation becomes

$$m^2 \approx \frac{(N^2 - \hat{\omega}^2)}{\hat{\omega}^2} k^2 = \frac{N^2}{\bar{u}^2} - k^2$$

- Short horizontal wavelengths (k > N/U) are vertically trapped.

Satellite image of vertically trapped mountain waves (courtesy of Sam Shen, U of Alta)

Momentum flux

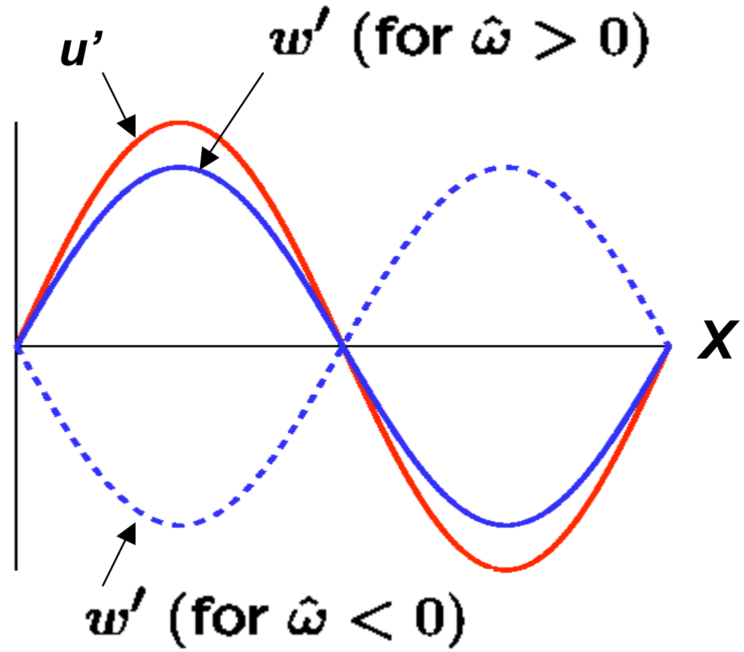
• vertically propagating gravity waves "transport" momentum over large height ranges.

• consider the continuity equation for the hydrostatic gravity in the case where *m* << 2*H*:

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} \approx 0$$

$$\Rightarrow \quad \tilde{w} = -\frac{k}{m}\tilde{u}$$

$$\therefore \quad \rho_0 \overline{u'w'} = \frac{1}{2}\,\rho_0 \frac{\hat{\omega}}{N}\,|\tilde{u}|^2$$



• GWs with eastward (westward) intrinsic frequencies have positive (negative) momentum flux.

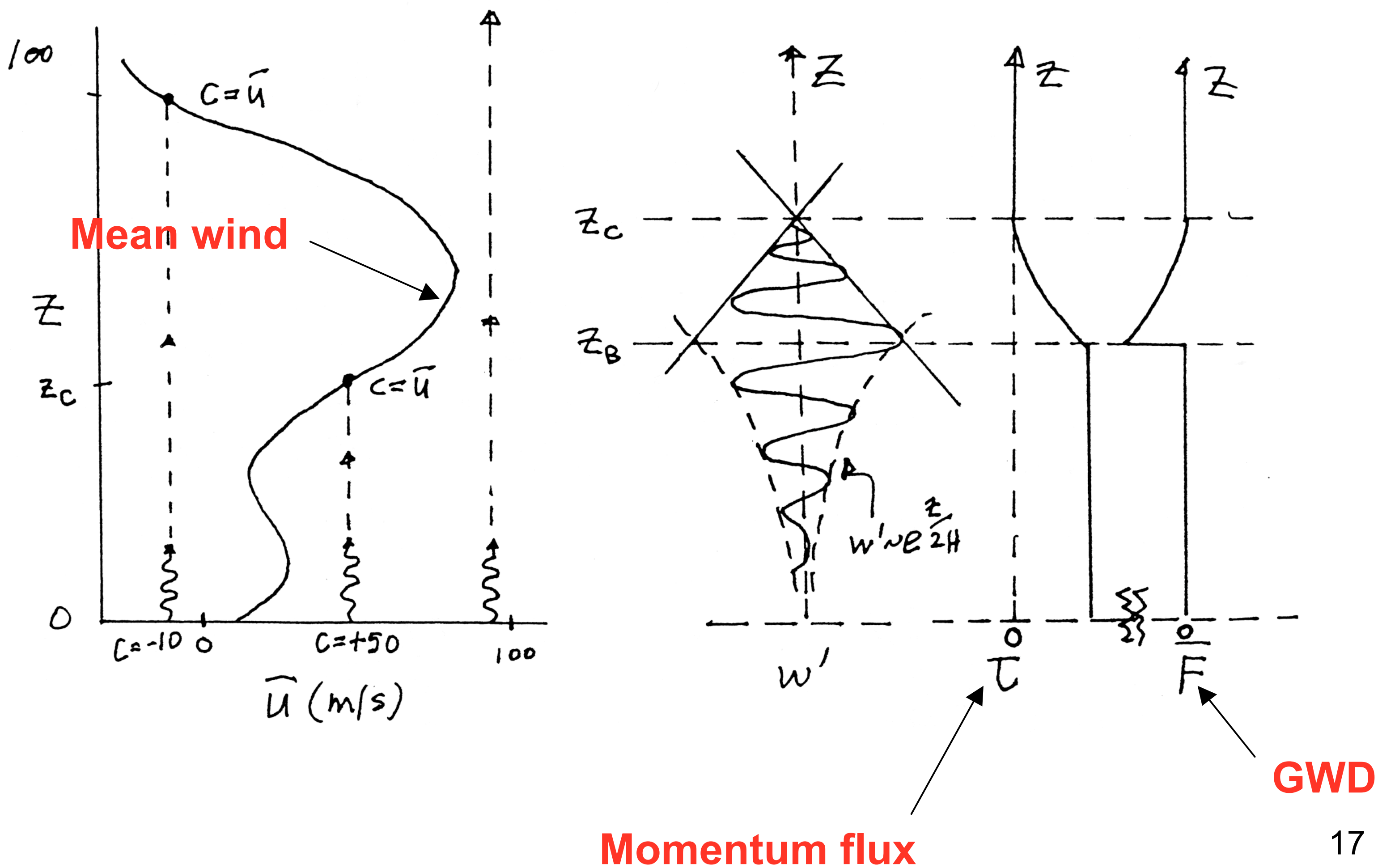
Momentum flux deposition

- gravity waves interact with the mean flow through the deposition of momentum:

$$\frac{\partial \bar{u}}{\partial t} + \ldots = \boxed{-\frac{1}{\rho_o}\frac{\partial}{\partial z}(\rho_o \overline{u'w'})} \longleftarrow \textbf{GWD}$$

- for a steady, linear & undamped GW, the momentum flux is independent of height $\Longrightarrow$ GWD = 0.
- GWD arises when the momentum flux changes with height, which will occur if:
  1. the GW approaches a critical level (c = U)
  2. the GW 'breaks' and undergoes turbulent dissipation.

Cartoon depicting critical level 'filtering' and nonlinear breakdown of a monochromatic GW



Mesoscale model simulations of convectively generated GWs

• to simulate small-scale GWs generated by convection a 2D or 3D mesoscale model must be used.

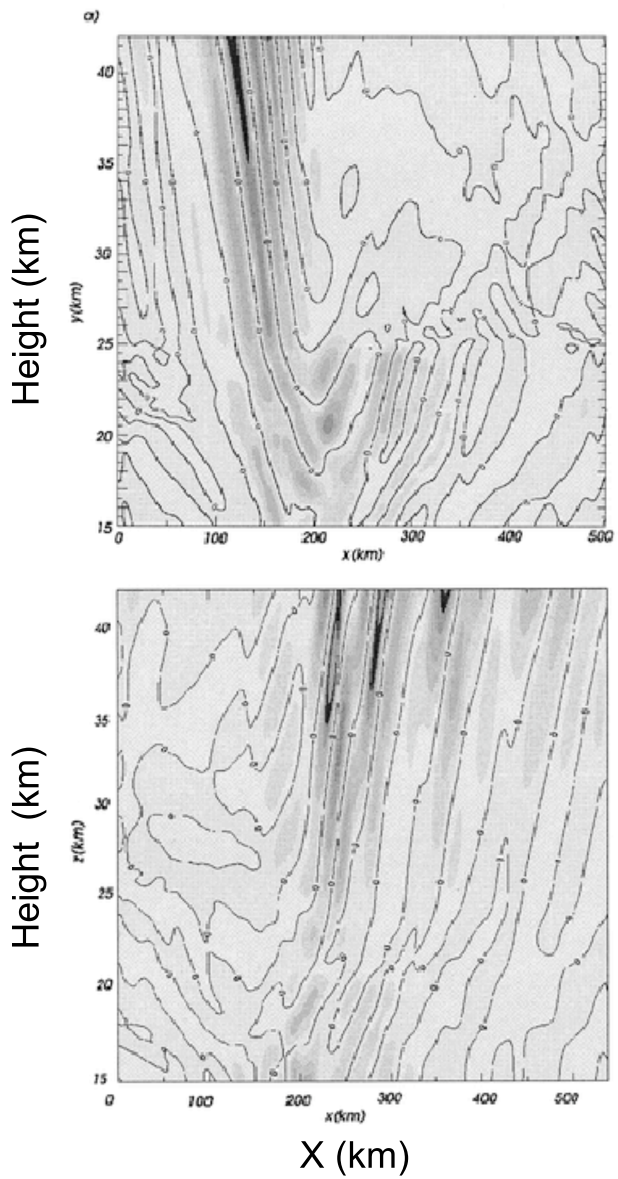
• compared to a GCM a mesoscale model employs a very fine horizontal resolution (1km) and a short timestep (5 s).

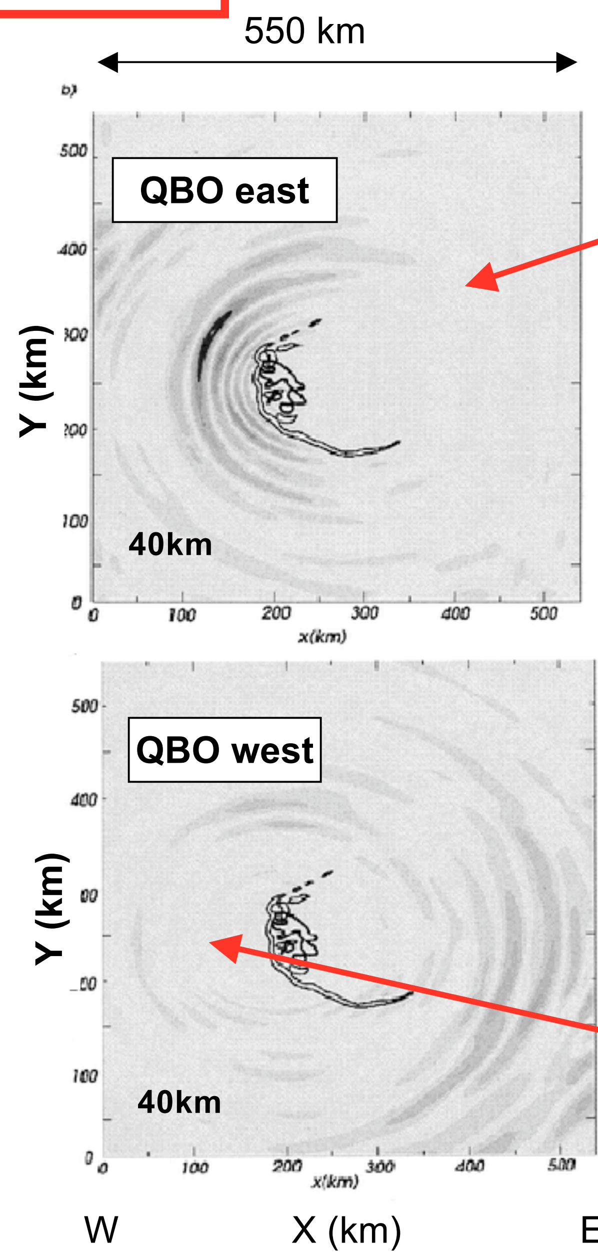
• here we will examine results from a 3D simulation of a tropical squall line.

• the model equations are nonlinear, compressible, nonhydrostatic and nonrotating; cloud microphysics parameterization is used.

3D mesoscale model results

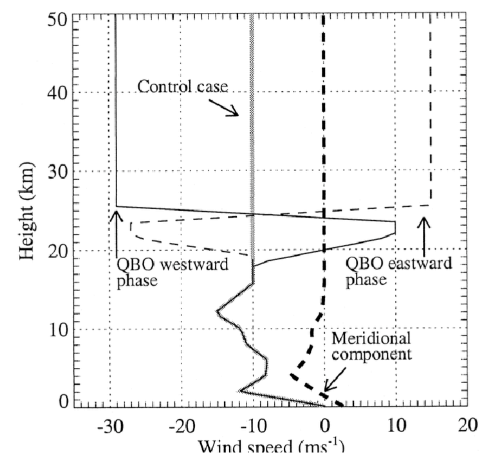
From Piani and Durran (JAS 2001)





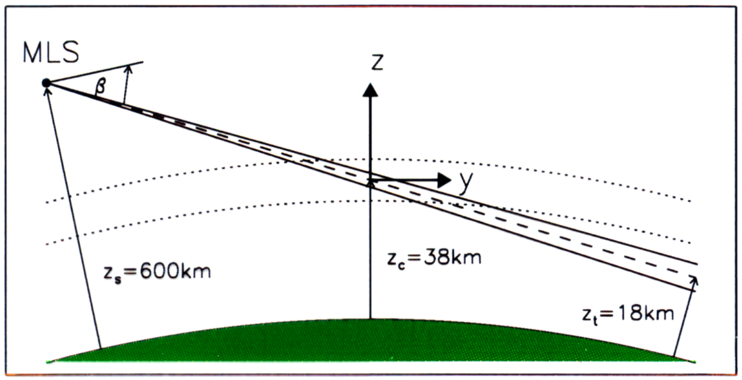
Eastward propagating GWs filtered out by eastward QBO winds

Westward propagating GWs filtered out by westward QBO winds

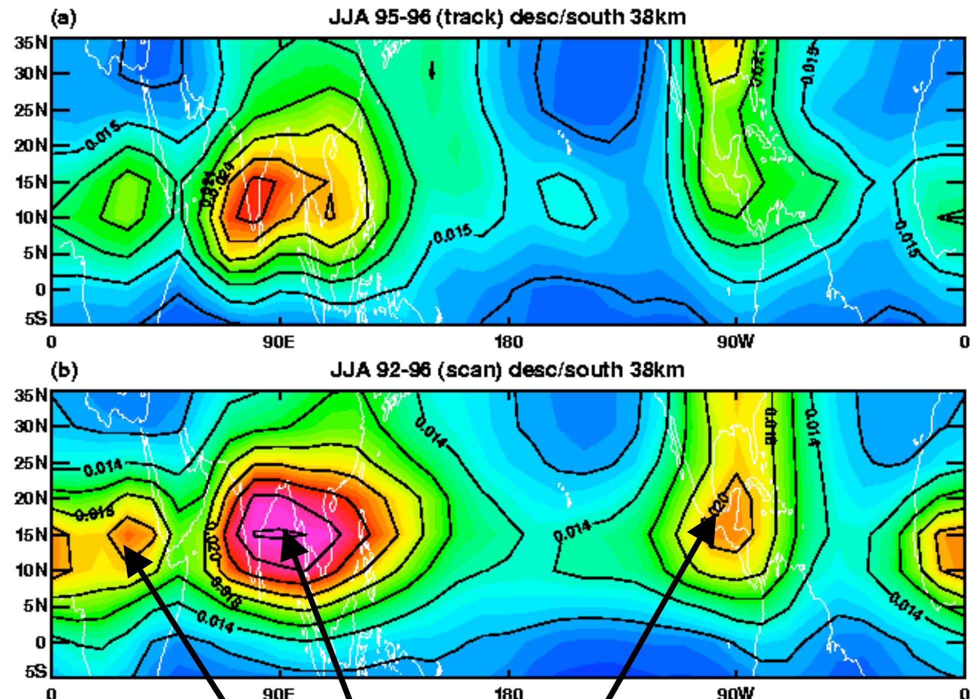


Satellite observations of convectively generated small-scale gravity waves

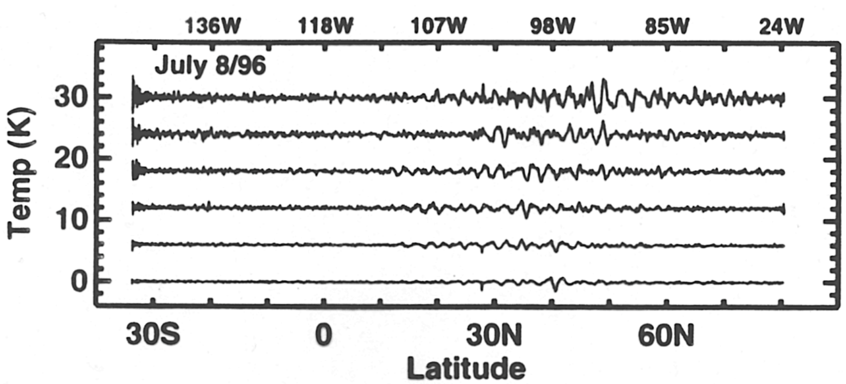
Microwave Limb Sounder geometry



Seasonal average (NH summer)



Single day of temperature data



McLandress et al (2000)

Temperature variance strongest over convective regions