3. Vorticity

3.1. *Definition and interpretation of vorticity*

The vorticity $\boldsymbol{\omega}$ is defined as the curl of the velocity field:

$$\boxed{\boldsymbol{\omega}(\boldsymbol{r}, t) = \nabla \times \boldsymbol{u}(\boldsymbol{r}, t)\ .}$$

Vorticity is a measure of the *local* 'spin' or 'rotation' of the fluid. (*Not* QM spin!) [See Acheson, pp. 11–12.]

For simplicity, consider 2D flow in $xy-$plane, $\partial/\partial z = 0$:

$$\boldsymbol{u} = \Big(u(x, y, t), v(x, y, t), 0\Big)\ ,$$

$$\text{so} \qquad \boldsymbol{\omega} = (0, 0, \omega) \qquad \text{where} \qquad \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\ .$$

Notes

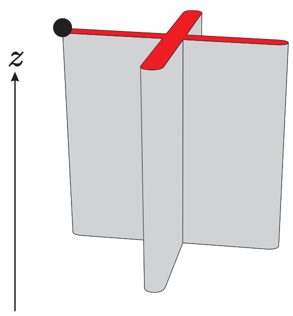
• For 'solid body rotation', $\boldsymbol{u} = \Omega r \boldsymbol{i}_\theta = (-\Omega y, \Omega x, 0)$ in Cartesians, $\boldsymbol{\omega} = (0, 0, 2\Omega)$.

• But $\boldsymbol{\omega}$ isn't generally a measure of large-scale rotation: e.g. consider a 'line vortex', with velocity $\propto \; 1/r$:

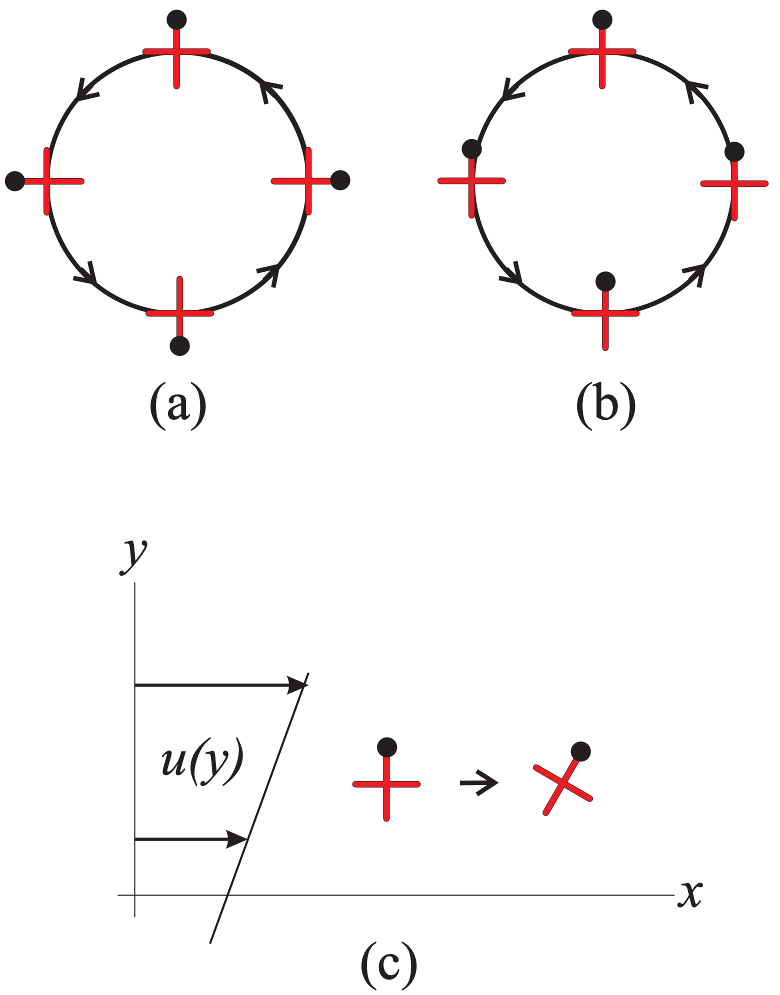
$$\boldsymbol{u} = \frac{A}{r}\boldsymbol{i}_\theta = \left(-\frac{Ay}{x^2+y^2}, \frac{Ax}{x^2+y^2}, 0\right) \; .$$

Here $\boxed{\boldsymbol{\omega} = \nabla \times \boldsymbol{u} = 0}$ (for $r \neq 0$). This is an example of *irrotational* (i.e. zero vorticity) flow, for $r \neq 0$.

• Imagine a hypothetical 'vorticity meter' placed in the flow:



(a) Solid body rotation: $|\boldsymbol{u}| \propto r$, flow gets faster as r increases, meter spins, $\Rightarrow \omega \neq 0$:



(b) Line vortex: $|\boldsymbol{u}| \propto r^{-1}$, flow gets slower as r increases, in just such a way that meter doesn't spin, $\Rightarrow \omega = 0$.

(c) Linear shear flow: $\boldsymbol{u} = (\beta y, 0, 0)$. No large-scale rotation, but $\omega = -\beta \neq 0$, $\Rightarrow$ local spin.

3.2. *The vorticity equation*

Starting with the Navier-Stokes equation, we can derive equations for the time-development of the vorticity.

Assume $\rho =$ const., incompressible, and $\nu = \mu/\rho =$ const.

N-S equation (1.10) $\Rightarrow$

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} &= -\frac{1}{\rho}\nabla p - g\boldsymbol{k} + \nu \nabla^2 \boldsymbol{u} \\ &= -\nabla\left(\frac{p}{\rho} + gz\right) + \nu\nabla^2\boldsymbol{u} \; . \end{aligned} \tag{3.1}$$

Incompressibility (1.9) $\Rightarrow$ $\nabla \cdot \boldsymbol{u} = 0$.

We need the vector identity

$$\boxed{(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = \tfrac{1}{2}\nabla|\boldsymbol{u}|^2 - \boldsymbol{u} \times (\nabla \times \boldsymbol{u})}$$

[check this: e.g. put $\boldsymbol{F} = \boldsymbol{G} = \boldsymbol{u}$ in formula for $\nabla(\boldsymbol{F} \cdot \boldsymbol{G})$ on Data Sheet].

So (3.1) $\Rightarrow$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \nabla\left(\tfrac{1}{2}|\boldsymbol{u}|^2 + \frac{p}{\rho} + gz\right) - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) = \nu \nabla^2 \boldsymbol{u}\,. \tag{3.2}$$

Now use the definition of vorticity, $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$. The curl of (3.2) gives

$$\boxed{\frac{\partial \boldsymbol{\omega}}{\partial t} - \nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = \nu \nabla^2 \boldsymbol{\omega}} \tag{3.3}$$

– this is called the *vorticity equation*.

[Uses curl(grad) $=$ 0 and $\nabla \times (\nabla^2 \boldsymbol{u}) = \nabla^2 (\nabla \times \boldsymbol{u})$.]

But $\nabla \cdot \boldsymbol{u} = 0$ and $\nabla \cdot \boldsymbol{\omega} = 0$ (since div(curl) $=$ 0); then a vector identity (see data sheet) $\Rightarrow$

$$\nabla \times (\boldsymbol{u} \times \boldsymbol{\omega}) = (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega}\,.$$

So (3.3) $\Rightarrow$

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \nu \nabla^2 \boldsymbol{\omega}$$

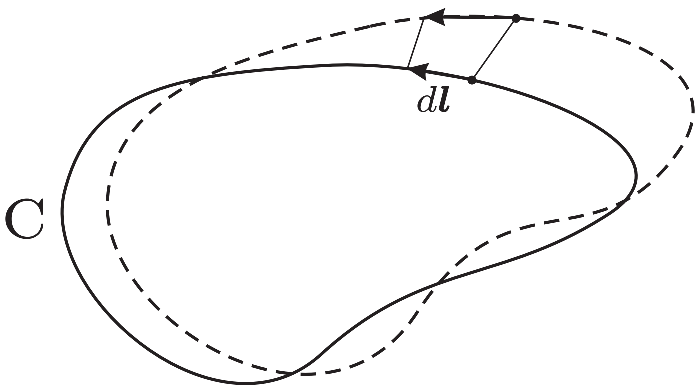
or

$$\boxed{\frac{D\boldsymbol{\omega}}{Dt} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u} = \nu \nabla^2 \boldsymbol{\omega}} \tag{3.4}$$

– another form of the vorticity equation.

3.3. *Kelvin's Circulation Theorem*

Suppose C is a closed circuit of fluid particles ('dyed') that moves with the flow, and S is any surface that spans C.



[The diagram shows circuit C at time t (solid) and at time $t + \delta t$ (dashed), and also a line element $d\boldsymbol{l}$ of C.]

The *circulation* around C is defined as

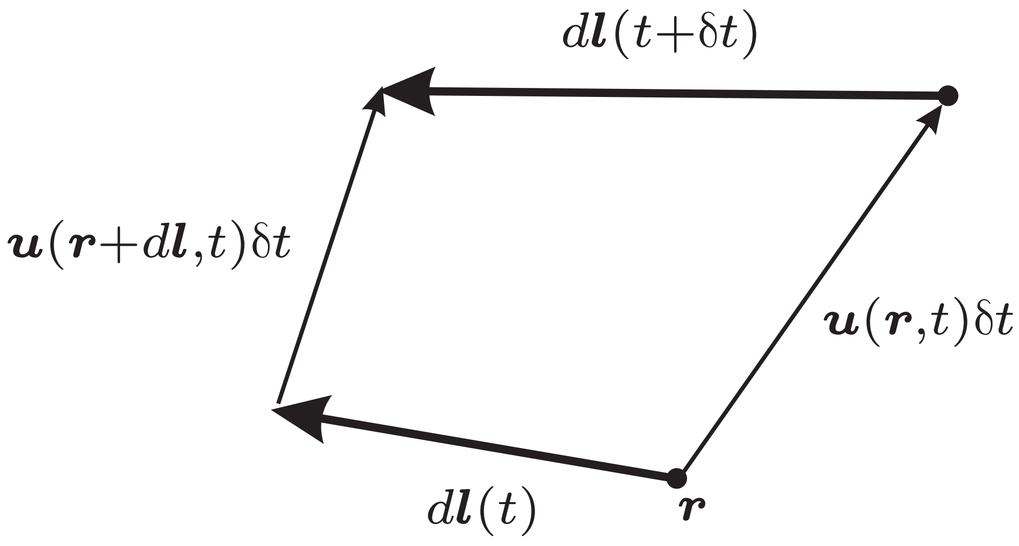
$$\boxed{\Gamma_{\mathrm{C}} \equiv \oint_{\mathrm{C}} \boldsymbol{u} \cdot d\boldsymbol{l}} \qquad \left(= \int_{\mathrm{S}} \boldsymbol{\omega} \cdot d\boldsymbol{S} \right) .$$

(The right-hand equality follows from Stokes's Theorem: see Problem 7.)

How does Γ_{C} change in time?

$$\frac{d\Gamma_{\mathrm{C}}}{dt} = \oint_{\mathrm{C}} \left(\frac{D\boldsymbol{u}}{Dt} \right) \cdot d\boldsymbol{l} + \oint_{\mathrm{C}} \boldsymbol{u} \cdot \left(\frac{D\, d\boldsymbol{l}}{Dt} \right) \qquad (3.5)$$

How does $d\boldsymbol{l}$ change in time? From diagram:



we have

$$\begin{aligned} d\boldsymbol{l}(t+\delta t) &\approx d\boldsymbol{l}(t) + \boldsymbol{u}(\boldsymbol{r}+d\boldsymbol{l},t)\delta t - \boldsymbol{u}(\boldsymbol{r},t)\delta t \\ &\approx d\boldsymbol{l}(t) + (d\boldsymbol{l}\cdot\nabla\boldsymbol{u})\,\delta t\ . \end{aligned}$$

So

$$\frac{D\,d\boldsymbol{l}}{Dt} \equiv \lim_{\delta t\to 0}\left(\frac{d\boldsymbol{l}(t+\delta t) - d\boldsymbol{l}(t)}{\delta t}\right) = d\boldsymbol{l}\cdot\nabla\boldsymbol{u}$$

and hence the second term on RHS of (3.5) becomes

$$\oint_{\mathrm{C}} \boldsymbol{u}\cdot(d\boldsymbol{l}\cdot\nabla\boldsymbol{u}) = \oint_{\mathrm{C}} d\boldsymbol{l}\cdot\nabla\left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) = \oint_{\mathrm{C}} d\left(\tfrac{1}{2}|\boldsymbol{u}|^2\right) = 0\ .$$

Substituting from NS, e.g. (3.1), into first term on RHS of (3.5), we get

$$\frac{d\Gamma_{\rm C}}{dt} = \oint_{\rm C} \left\{ -\nabla \left(\frac{p}{\rho} + gz \right) + \nu \nabla^2 \boldsymbol{u} \right\} \cdot d\boldsymbol{l} \ .$$

Again the $\nabla(...) \cdot d\boldsymbol{l}$ term integrates out, leaving

$$\begin{aligned} \frac{d\Gamma_{\rm C}}{dt} &= \nu \oint_{\rm C} (\nabla^2 \boldsymbol{u}) \cdot d\boldsymbol{l} \\ &= 0 \quad \text{if inviscid.} \end{aligned} \tag{3.6}$$

Eq.(3.6) is *Kelvin's Circulation Theorem*: for inviscid, uniform-density flow, the circulation is constant following a closed moving material circuit.

This is often useful, for both conceptual and practical purposes.

3.4. *Conservation of vorticity in 2D inviscid flow*

Inviscid: put $\nu = 0$ in (3.4). (Re $= UL/\nu \to \infty$.)

Consider 2D flow again, $\boldsymbol{u} = (u, v, 0)$, $\boldsymbol{\omega} = (0, 0, \omega)$. Then $\boldsymbol{\omega} \cdot \nabla = \omega \partial/\partial z = 0$, so just get

$$\boxed{\frac{D\omega}{Dt} = 0\ .}$$

Interpretation: vorticity of each fluid blob is conserved *following the motion of the blob*.

In particular, if the fluid is irrotational everywhere at $t = 0$, then it remains irrotational for $t > 0$.

$$\boxed{\Rightarrow \textbf{inviscid, irrotational flow.}}$$

This was a favourite topic for 19th century applied mathematicians: see, e.g., Lamb's *Hydrodynamics* ($>$ 700 pages). But it's still a useful idealisation for modelling many important physical phenomena.